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Reply by Author to J. D. Revell

A. M. CUNNINGHAM JR*

General Dynamics Corporation, Fort Worth Division, Fort Worth, Texas

THE author wishes to thank Revell for pointing out areas which require further discussion. These areas are concerned with the relationship of the present method with other methods, types of pressure functions, and the minimum number of downwash points (or pressure functions) required for solutions of acceptable accuracy.

The purpose of the research partly summarized in this paper is to develop an aerodynamic prediction technique which combines the efficiency of Hsu's method with the accuracy of the NASA method[1] or Laschka's method.[2] In particular, this paper is concerned with the instabilities of Hsu's method and the refinements that are necessary to remove them. This paper also represents the first of two papers which, together, present a method of accurately predicting airloads on arbitrary planar surfaces in steady or oscillatory subsonic flow with minimum computational effort.

As was pointed out in the summary, introduction, and conclusions, the paper had several other major objectives besides discussing the importance of chordwise integration and developing a more accurate means of performing this step. One of these was to establish a criterion for choosing, a priori, a minimum number of downwash points, and, hence, pressure functions, that would yield a solution of acceptable accuracy. Another objective was to show that the present method was more efficient than finite-element methods for predicting aerodynamic loadings on planar surfaces of arbitrary planform.

The present method is compared with several other classic methods in this paper and is shown to be equal or superior in accuracy. In a following paper[3] on the oscillatory version of this method, it is also shown to be equal in accuracy to Laschka's method on planar surfaces without discontinuities in planform geometry. This constitutes an indirect comparison with the NASA method developed by Watkins, Woolston, and H. J. Cunningham,[2] since Laschka has shown that his method yields essentially same results as NASA method.

The pressure functions used in the present solution, however, are not the best. For the chordwise functions, it is felt by the author that those used by Laschka[2] and Wagner[4] might be the best. For the spanwise functions, the present method has been modified to use Tschebyshev polynomials of the second kind $U_n(\mathbf{x})$ as defined in Ref. 5, in lieu of the simple polynomials as given in Eq. (3b). This modification yields matrices which are far better behaved than those obtained for the simple polynomial type of spanwise functions. As a result, the method has been applied successfully to surfaces with strong discontinuities in the local chord such as those encountered in planar representations of wing-body or wing-tip store configurations.

The most important aspect questioned by Revell is the choice of a suitable number of downwash points and the corresponding pressure functions. The basic question here is how well the chosen downwash points define the true downwash distribution. This all depends on the spacing of the points in both the chordwise and spanwise directions.

If an even spacing of points is assumed, this effectively results in a definition of the downwash at all points between the control points with a Taylor Series expansion of the distribution. Such expansions have the slowest possible convergence,[5] and the resulting interpolation functions can diverge even for very smooth downwash functions.[6] Thus the use of evenly spaced downwash points can lead to pressure function solutions whose induced downwash functions oscillate violently between the downwash points. In this case, a least-squares approach would provide some stability, but it could never yield the best fit, which is unique.[6]

If an uneven spacing is used, specifically the $\mathbf{x}_i$ as given in Eq. (19) in the chordwise direction, the quality of the curve fitting is vastly improved. The use of these points yields an interpolated function with accuracy and smoothness that approaches the best fit possible in the least-squares sense without the difficulties inherent in arbitrarily choosing a least-squares solution. The $\mathbf{x}_i$, for $i = 1, 2, \ldots, \bar{m}$, are the $\bar{m}$ zeros of a polynomial $Q_{\bar{m}}(\mathbf{x})$ defined as $Q_0(\mathbf{x}) = 1$, $Q_{\bar{m}}(\mathbf{x}) = [U_{\bar{m}}(\mathbf{x}) - U_{\bar{m}+1}(\mathbf{x})]$, $\bar{m} > 0$, where $U_n(x)$ are Tschebyshev polynomials of the second kind. The $\mathbf{x}_i$ were originally chosen by Hsu to yield zero error in the predicted sectional lift if the chordwise downwash variation is of degree $(2\bar{m} - 1)$ or less. [The generalized force is exact if the downwash is of degree $(\bar{m} - 1)$ or less.] This consideration was made for average lift; however, something can also be said for what happens over the entire surface. Since the set of $\mathbf{x}_i$ is the zeros of $Q_m(x)$, then it can be shown that the error in the induced downwash, $\Delta w(\mathbf{x})$, for $\bar{m} \geq 3$ is approximately given as $\Delta w(x) \approx a_{\bar{m}} U_{\bar{m}}(\mathbf{x}) + a_{\bar{m}+1} U_{\bar{m}+1}(\mathbf{x}) + a_{\bar{m}+2} U_{m+2}\mathbf{x}) + \ldots$, where the a_n are the coefficients of the $n \geq \bar{m}$ terms of a $U_n(\mathbf{x})$ polynomial expansion of the exact downwash function. Thus one may establish approximate error bounds on the induced downwash at points other than the control points by the $\Delta w(\mathbf{x})$ given previously. Although a $U_n(\mathbf{x})$ expansion does not converge quite as rapidly as an expansion of Tschebyshev polynomials of the first kind, $T_n(\mathbf{x})$, it is far more rapid than a Taylor series expansion.

As an estimate of the error involved, the ratio of the $a_{\bar{m}}$ coefficient from the previous equation to the $b_{\bar{m}}$ coefficient of a Taylor series expansion of the same function is approximately[5] $a_{\bar{m}}/b_{\bar{m}} \approx (\bar{m} + 1)/2^{\bar{m}}$, Thus for $\bar{m} = 4$, $a_4/b_4 = \frac{5}{16}$; and for $\bar{m} = 5$, $a_5/b_5 = \frac{6}{32}$. For the $c_{\bar{m}}$ coefficient of a $T_n(x)$ expansion of the same function, the ratio is $c_{\bar{m}}/b_{\bar{m}} \approx 1/2^{\bar{m}-1}$, which yields for $\bar{m} = 4$, $c_{\bar{m}}/b_{\bar{m}} = \frac{1}{8}$, and for $\bar{m} = 5$, $c_{\bar{m}}/b_{\bar{m}} = \frac{1}{16}$. Thus if the degree of the camber is known, then an almost exact fit may be achieved. If the degree is not known, then, because of rapid convergence of $U_n(\mathbf{x})$ expansion, the fit will approach best fit as afforded by $T_n(\mathbf{x})$ expansion.

More specifically, experience has shown that it is sufficient to use a number of chordwise points, or pressure functions, that are a minimum of three or equal to two plus the estimated degree of the camber line variation. Then by use of the ratio defined by Eq. (24) to determine the number of spanwise terms, pressure distributions are predicted which show excellent correlation with local measured distributions.

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* Senior Structures Engineer. Member AIAA.

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